## Problem 5

A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in and then released, and if there is no damping, determine the position u of the mass at any time t. Plot u versus t. Find the frequency, period, and amplitude of the motion.

Solution

Start by drawing a free-body diagram of the mass. The two forces acting on it are due to the spring and gravity.
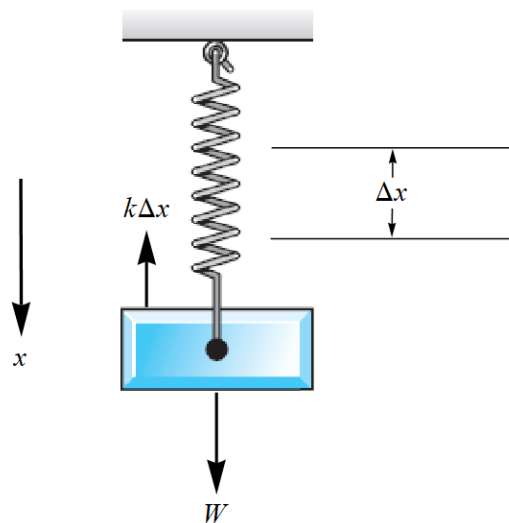
If the mass is stationary and hanging from the spring, the gravitational and spring forces balance each other.

$$W = k\Delta x$$

From this equation, k can be determined.

$$2 \text{ lb} = k(6 \text{ in})$$

$$k = \frac{1}{3}\frac{\text{lb}}{\text{in}} \times \frac{12 \text{ in}}{1 \text{ ft}} = 4\,\frac{\text{lb}}{\text{ft}}$$

Now we will apply Newton's second law in the x-direction to obtain the equation of motion for the mass.

$$\sum F_x = ma_x$$

$$-kx + W = ma_x$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$-kx + W = mx''$$

$$mx'' + kx = W$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$mx_c'' + kx_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad x_c' = re^{rt} \quad \rightarrow \quad x_c'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$m(r^2 e^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + k = 0$$

$$r^2 = -\frac{k}{m}$$

$$r = \pm i\sqrt{\frac{k}{m}} = \pm i\omega$$

Two solutions to equation (1) are then

$$x_c = e^{-i\omega t} \quad \text{and} \quad x_c = e^{i\omega t}.$$

By the principle of superposition, the general solution for x_c is a linear combination of these two.

$$\begin{aligned} x_c(t) &= C_1 e^{-i\omega t} + C_2 e^{i\omega t} \\ &= C_1[\cos(-\omega t) + i\sin(-\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\ &= C_1[\cos(\omega t) - i\sin(\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\ &= C_1 \cos\omega t - iC_1 \sin\omega t + C_2 \cos\omega t + iC_2 \sin\omega t \\ &= (C_1 + C_2)\cos\omega t + (-iC_1 + iC_2)\sin\omega t \\ &= C_3 \cos\omega t + C_4 \sin\omega t \end{aligned}$$

On the other hand, the particular solution satisfies

$$mx_p'' + kx_p = W.$$

Because the inhomogeneous term is a constant, the particular solution is a constant as well: $x_p(t) = A$. Substitute this into the equation to determine A.

$$m(A)'' + k(A) = W$$

$$kA = W$$

$$A = \frac{W}{k}$$

So then $x_p(t) = W/k$, which means that the general solution for x is

$$x(t) = C_3 \cos\omega t + C_4 \sin\omega t + \frac{W}{k}.$$

Take a derivative of it with respect to t.

$$x'(t) = -C_3\omega \sin\omega t + C_4\omega \cos\omega t$$

Now apply the initial conditions,

$$x(0) = 6 \text{ in} + 3 \text{ in} = 9 \cancel{\text{in}} \times \frac{1 \text{ ft}}{12 \cancel{\text{in}}} = \frac{3}{4} \text{ ft}$$

$$x'(0) = 0 \frac{\text{ft}}{\text{s}},$$

to determine C_3 and C_4.

$$x(0) = C_3 + \frac{W}{k} = \frac{3}{4}$$

$$x'(0) = C_4 \omega = 0$$

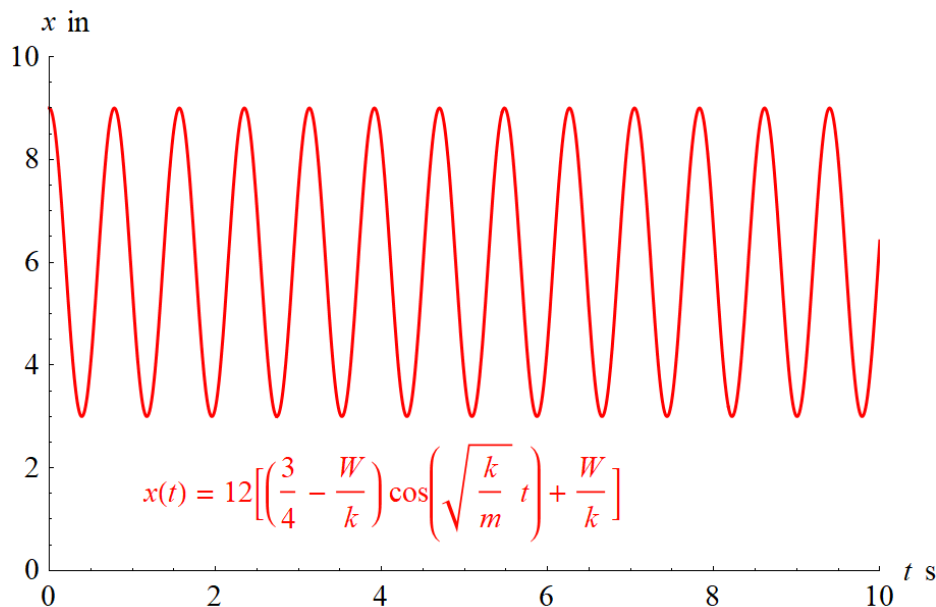Solving this system of equations yields

$$C_3 = \frac{3}{4} - \frac{W}{k} \quad \text{and} \quad C_4 = 0,$$

so

$$x(t) = \left(\frac{3}{4} - \frac{W}{k} \right) \cos \omega t + \frac{W}{k}$$

$$= \left(\frac{3}{4} - \frac{W}{k} \right) \cos \left(\sqrt{\frac{k}{m}} t \right) + \frac{W}{k}.$$

Finally, plug in the numbers, $W = 2$ lb and $k = 4$ lb/ft. The mass m is obtained by dividing the weight by the gravity: $m = W/g = 2 \text{ lb}/(32.2 \text{ ft/s}^2)$. As $x(t)$ is in feet, multiply the result by 12 to convert it to inches.

The amplitude is $(3/4 - W/k) = 1/4$ ft $= 3$ in, the angular frequency is $\omega = \sqrt{k/m} \approx 8.02$ rad/s, and the period is $T = 2\pi/\omega \approx 0.783$ s.